The characterizations of the stable perturbation of a closed operator by a linear operator in Banach spaces

Fapeng Du*
School of Mathematical & Physical Sciences, Xuzhou Institute of Technology
Xuzhou 221008, Jiangsu Province, P.R. China

Yifeng Xue†
Department of mathematics, East China Normal University
Shanghai 200241, P.R. China

Abstract

In this paper, we investigate the invertibility of $I_Y + \delta T T^+$ when T is a closed operator from X to Y with a generalized inverse T^+ and δT is a linear operator whose domain contains $D(T)$ and range is contained in $D(T^+)$. The characterizations of the stable perturbation $T + \delta T$ of T by δT in Banach spaces are obtained. The results extend the recent main results of Huang's in Linear Algebra and its Applications.

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1 Introduction

The expression and perturbation analysis of the generalized inverse (resp. the Moore–Penrose) inverse of bounded linear operators on Banach spaces (resp. Hilbert spaces) have been widely studied since Nashed's book [10] was published in 1976. Ten years ago, Chen and Xue proposed a notation so–called the stable perturbation of a bounded operator instead of the rank–preserving perturbation of a matrix in [2]. Using this new notation, they established the perturbation analyses for the Moore–Penrose inverse and the least square problem on Hilbert spaces in [3], [5], [6] and [12]. In recent years, the perturbation analysis of generalized inverses of closed operators has been appeared in [7], [8] and [11] with small perturbation operators bounded related to closed operators. The results in these papers generalize corresponding results in [2].

Throughout the paper, X and Y are always Banach spaces. Let $B(X, Y)$, $D(X, Y)$ and $C(X, Y)$ denote the set of bounded linear operators, densely–defined linear operators from X to Y and closed densely–defined linear operators from X to Y, respectively. For $T \in D(X, Y)$, let $R(T)$ (resp. $N(T)$) denote the range (resp. null space) of T. Suppose that $T \in C(X, Y)$ has a generalized inverse T^+. Let $\delta T\colon D(\delta T) \to Y$ be a closed operator with $D(T) \subset D(\delta T)$ and $R(\delta T) \subset D(T^+)$. Put

*E-mail: jsdfp@163.com
†Corresponding author, E-mail: yfxue@math.ecnu.edu.cn

$\bar{T} = T + \delta T$. In this paper, we first characterize when $I_Y + \delta T T^+ \colon D(T^+) \to D(T^+)$ is bijective and then give some equivalent conditions that make $R(\bar{T}) \cap N(T^+) = \{0\}$ under the assumption that $I_Y + \delta T T^+ \colon D(T^+) \to D(T^+)$ is bijective. These results generalize several main results in [7, 8].

2 Some Lemmas

Let V be a closed subspace of X. Recall that V is complemented in X if there is a closed subspace U in X such such $V \cap U = \{0\}$ and $X = V + U$. In this case, we set $X = V \dotplus U$ and $U = V^c$.

Let $T \in B(X, Y)$. If there is $S \in B(Y, X)$ such that $TST = T$ and $STS = S$, then we say T has a generalized inverse S, denoted by T^+. It is well–known that $T \in B(X, Y)$ has a $T^+ \in B(Y, X)$ iff $R(T)$ is closed and

$$X = N(T) \dotplus N(T)^c, \quad Y = R(T) \dotplus R(T)^c$$

(cf. [4]). In general, we have

Definition 2.1. *Let $T \in C(X, Y)$. If there is $S \in D(Y, X)$ with $D(S) \supset R(T)$ and $R(S) \subset D(T)$ such that*

$$TST = T \text{ on } D(T), \quad STS = S \text{ on } D(S), \tag{2.1}$$

then S is called a generalized inverse of T, denoted by T^+.

From (2.1), we get that $P = I_X - ST$ (resp. $Q = TS$) is an idempotent operator on $D(T)$ (resp. $D(S)$) with $R(P) = N(T)$ (resp. $R(Q) = R(T)$). Let $T \in C(X, Y)$. It is known that for $T \in C(X, Y)$, we can always find a $T^+ \in D(Y, X)$ (cf. [10]) and we call T^+ is an algebraic generalized inverse of T. But when T^+ becomes a closed operator is a problem. The following proposition (cf. [10]) gives an answer.

Proposition 2.2. *Let $T \in C(X, Y)$. Assume that $Y = \overline{R(T)} \dotplus (\overline{R(T)})^c$. Let $Q \colon Y \to \overline{R(T)}$ be the bounded idempotent operator on Y.*

(1) *If there is a closed subspace M of X such that $M \cap N(T) = \{0\}$ and $D(T) = N(T) + M \cap D(T)$, then $T^+ \in C(Y, X)$ with $D(T^+) = R(T) + (\overline{R(T)})^c$, $R(T^+) = D(T) \cap M$ and $TT^+ y = Qy$, $\forall\, y \in D(T^+)$.*

(2) *If $X = N(T) \dotplus N(T)^c$, then there exists a unique $S \in C(Y, X)$ with $D(S) = R(T) + (\overline{R(T)})^c$, $N(S) = (\overline{R(T)})^c$ and $R(S) = D(T) \cap N(T)^c$ such that*

$$TST = T \text{ on } D(T) \text{ and } STS = S \text{ on } D(S) \tag{2.2}$$

$$TS = Q \text{ on } D(S) \text{ and } \quad ST = I_X - P \text{ on } D(T), \tag{2.3}$$

where P is the idempotent operator of X onto $N(T)$.

In addition, S is bounded if $R(T)$ is closed.

Proof.(1) Put $A = T\big|_{M \cap D(T)}$. It is easy to check that A is a closed operator with $N(A) = \{0\}$ and $R(A) = R(T)$. Thus, $A^{-1} \colon R(T) \to M \cap D(T)$ is also a closed

operator. Set $Sy = \begin{cases} A^{-1}y & y \in R(T) \\ 0 & y \in (\overline{R(T)})^c \end{cases}$. Then $D(S) = R(T) + (\overline{R(T)})^c$ is dense in Y, $R(S) = M \cap D(T) \subset D(T)$ and

$$TST = T \text{ on } D(T), \quad STS = S \text{ on } D(S), \ TS = Q \text{ on } D(S).$$

To show that $S \in C(Y, X)$, let $\{y_n\}_{n=1}^{\infty} \subset D(S)$ such that $\|y_n - y\| \to 0$ and $\|Sy_n - x\| \to 0$ as $n \to \infty$ for some $y \in Y$ and $x \in X$. Note that $Qy_n \in R(T)$, $Sy_n = SQy_n = A^{-1}Qy_n$, $n \geq 1$ and $\|Qy_n - Qy\| \to 0$. Since $A^{-1} \in C(\overline{R(T)}, X)$, it follows that $Qy \in R(T)$ and $A^{-1}Qy = x$ and consequently, $y = Qy + (I_Y - Q)y \in D(S)$ and $Sy = SQy = x$. Thus, $S \in C(Y, X)$.

(2) Let $M = N(T)^c$ in (1). Then by the proof of (1), S satisfies the requirements of Proposition 2.2 (2).

Assume that there is another $S' \in C(Y, X)$ with $D(S') = R(T) + (\overline{R(T)})^c$ such that S' satisfies (2.2) and (2.3). Then

$$S' = S'TS' = (I_X - P)S' = STS' = SQ = STS = S \text{ on } D(S).$$

When $R(T)$ is closed, $D(S) = Y$. So S is bounded by Closed Graph Theorem.

The operator S in Proposition 2.2 (2) is denoted by $T^+_{P,Q}$.

Let H and K be Hilbert spaces. For a closed subspace M in H (or K), let P_M denote the orthogonal projection from H (or K) to M. According to Proposition 2.2 and its proof, we have

Corollary 2.3. *Let $T \in C(H, K)$. Then there is a unique $S \in C(K, H)$ with $D(S) = R(T) + R(T)^{\perp}$ and $R(S) = N(T)^{\perp} \cap D(T)$ such that*

$$TST = T \text{ on } D(T) \qquad \text{and } STS = S \text{ on } D(S)$$
$$TSy = P_{\overline{R(T)}}y, \ \forall\, y \in D(S) \ \text{ and } STx = P_{\overline{N(T)^{\perp} \cap D(T)}}x, \ \forall\, x \in D(T).$$

In addition, if $R(T)$ is closed, then S is bounded.

The operator S in Corollary 2.3 is called the the maximal Tseng inverse of T (cf. [1]), denote by $T^{\dagger}$. Clearly, $N(T^{\dagger}) = R(T)^{\perp}$ and $R(T^{\dagger}) = N(T)^{\perp} \cap D(T)$.

Lemma 2.4. *Let $T \in C(X, Y)$ with $T^+ \in D(Y, X)$ and let $\delta T \colon D(\delta T) \subset X \to Y$ be a linear operator with $D(T) \subset D(\delta T)$. Put $\bar{T} = T + \delta T$. If $R(\delta T) \subset D(T^+)$, then $I_Y + \delta T T^+ \colon D(T^+) \to D(T^+)$ is bijective if and only if $I_X + T^+\delta T \colon D(T) \to D(T)$ is bijective.*

Proof. Suppose that $I_Y + \delta T T^+$ is bijective. Then there is an operator $C \colon D(T^+) \to D(T^+)$ such that $(I_Y + \delta T T^+)C = C(I_Y + \delta T T^+) = I_Y$ on $D(T)$, that is,

$$C\delta T T^+ = \delta T T^+ C = I_X - C \text{ on } D(T^+). \tag{2.4}$$

Thus, from (2.4), we get that for any $\xi \in D(T^+)$,

$$\begin{aligned}(I_X + T^+\delta T)(I_X - T^+C\delta T)\xi &= \xi + T^+\delta T\xi - T^+C\delta T\xi - T^+\delta T T^+ C\delta T\xi \\ &= \xi + T^+\delta T\xi - T^+C\delta T\xi - T^+(I_X - C)\delta T\xi \\ &= \xi.\end{aligned}$$

Similarly, $(I_X - T^+C\delta T)(I_X + T^+\delta T)\xi = \xi$, $\forall\, \xi \in D(T^+)$. Therefore, $I_X + T^+\delta T$ is bijective.

Conversely, if $I_X + T^+\delta T$ is bijective, we can obtain that $I_Y + \delta TT^+$ by using similar way.

Lemma 2.5. *Let $T \in C(X, Y)$ with $T^+ \in D(Y, X)$. Let $\delta T\colon D(\delta T) \subset X \to D(T^+)$ be a linear operator such that $D(T) \subset D(\delta T)$. Put $\bar{T} = T + \delta T$. Assume that $I_X + T^+\delta T : D(T) \to D(T)$ is bijective and $R(\bar{T}) \cap N(T^+) = \{0\}$. Then $N(\bar{T}) = (I_X + T^+\delta T)^{-1}N(T)$.*

Proof. Let $x \in N(\bar{T})$. Then $Tx = -\delta Tx$ and $(I_X - T^+T)x = (I_X + T^+\delta T)x$. Note that $(I_X - T^+T)x \in N(T)$ and $I_X + T^+\delta T$ is bijective. So $x \in (I_X + T^+\delta T)^{-1}N(T)$.

Now let $x \in N(T)$ and put $z = (I_X + T^+\delta T)^{-1}x$. Then $(I_X + T^+\delta T)z = x$ and $T(I_X + T^+\delta T)z = 0$. Thus, $T^+\bar{T}z = 0$. Since $R(\bar{T}) \cap N(T^+) = \{0\}$, it follows that $\bar{T}z = 0$, i.e., $x \in N(\bar{T})$.

3 Stable perturbation in Banach spaces

Let $T \in C(X, Y)$ and let $\delta T\colon D(\delta T) \to Y$ be a linear operator with $D(T) \subset D(\delta T)$. Recall that δT is T–bounded if there are constants a, $b > 0$ such that

$$\|\delta Tx\| \le a\|x\| + b\|Tx\|, \quad \forall\, x \in D(T).$$

We have known from [9, Chap 4, Theorem 1.1] that $\bar{T} = T + \delta T \in C(X, Y)$ when δT is T–bounded with $b < 1$.

Let $T \in C(X, Y)$ such that T^+ exists and let $\delta T\colon D(\delta T) \to Y$ be a linear operator with $D(T) \subset D(\delta T)$, T–bounded and $b < 1$. Put $\bar{T} = T + \delta T \in C(X, Y)$. According to [2], we say $\bar{T}$ is a stable perturbation of T if $R(\bar{T}) \cap N(T^+) = \{0\}$.

The following theorem characterizes when $I_Y + \delta TT^+\colon D(T^+) \to D(T^+)$ is bijective and $\bar{T}$ is a stable perturbation of T.

Theorem 3.1. *Let $T \in C(X, Y)$ with $T^+ \in D(Y, X)$ and let $\delta T\colon D(\delta T) \to D(T^+)$ be a linear operator such that $D(T) \subset D(\delta T)$ and δT is T–bounded with $b < 1$. Put $\bar{T} = T + \delta T \in C(X, Y)$. Then the following statements are equivalent:*

(1) $I_Y + \delta TT^+\colon D(T^+) \to D(T^+)$ *is bijective*;

(2) $T^+\bar{T}\big|_{R(T^+)} = (I_X + T^+\delta T)|_{R(T^+)}\colon R(T^+) \to R(T^+)$ *is bijective*;

(3) $D(T^+) = \bar{T}R(T^+) + N(T^+)$, $\bar{T}R(T^+) \cap N(T^+) = \{0\}$ *and* $N(\bar{T}) \cap R(T^+) = \{0\}$.

Proof. (1)⇒(2) Assume that $W = I_Y + \delta TT^+\colon D(T^+) \to D(T^+)$ is bijective. From $W = \bar{T}T^+ + (I_Y - TT^+)$ and $(I_Y - TT^+)D(T^+) = N(T^+)$, we get that $D(T^+) = WD(T^+) \subset \bar{T}R(T^+) + N(T^+)$. Note that $\bar{T}R(T^+) \subset D(T^+)$ and $N(T^+) \subset D(T^+)$. So $\bar{T}R(T^+) + N(T^+) = D(T^+)$ and consequently, $R(T^+) = T^+\bar{T}R(T^+)$. This shows that $D = T^+\bar{T}\big|_{R(T^+)}\colon R(T^+) \to R(T^+)$ is surjective.

Now let $\xi \in R(T^+)$ and $T^+\bar{T}\xi = 0$. Then

$$(I_X + T^+\delta T)\xi = (I_X - T^+T)\xi + T^+\bar{T}\xi = 0$$

and consequently, $\xi = 0$ by Lemma 2.4, that is, D is injective.

Noting that $T^+\bar{T}T^+ = T^+(T+\delta T)T^+ = (I_X + T^+\delta T)T^+$, we have $D = (I_X + T^+\delta T)\big|_{R(T^+)}$.

(2)$\Rightarrow$(3) For any $\xi \in D(T^+)$ there is $\eta \in D(T^+)$ such that $T^+\xi = T^+\bar{T}T^+\eta$ since D is surjective. Thus, $\zeta = \xi - \bar{T}T^+\eta \in N(T^+)$ and so that $D(T^+) \subset \bar{T}R(T^+) + N(T^+) \subset D(T^+)$.

Let $\xi \in \bar{T}R(T^+) \cap N(T^+)$. Then $T^+\xi = 0$ and $\xi = \bar{T}T^+\eta$ for some $\eta \in D(T^+)$. So $DT^+\eta = 0$. Since D is injective, we have $T^+\eta = 0$ and so that $\xi = 0$. This proves that $\bar{T}R(T^+) \cap N(T^+) = \{0\}$.

Similarly, we can obtain $N(\bar{T}) \cap R(T^+) = \{0\}$.

(3)$\Rightarrow$(1) Since $D(T^+) = \bar{T}R(T^+) + N(T^+)$, it follows that for any $\eta \in D(T^+)$, there is $\xi_1 \in D(T^+)$ and $\xi_2 \in N(T^+)$ such that $\eta = \bar{T}T^+\xi_1 + \xi_2$. Put $\xi = TT^+\xi_1 + \xi_2 \in D(T^+)$. Then

$$(I_Y + \delta TT^+)\xi = (I_Y - TT^+)\xi + \bar{T}T^+\xi = \xi_2 + \bar{T}T^+\xi_1 = \eta,$$

that is, $I_Y + \delta TT^+ \colon D(T^+) \to D(T^+)$ is surjective.

To prove $I_Y + \delta TT^+$ is injective, let $\zeta \in D(T^+)$ such that $(I_Y + \delta TT^+)\zeta = 0$. Then $(I_Y - TT^+)\zeta = -\bar{T}T^+\zeta$. Since $\bar{T}R(T^+) \cap N(T^+) = \{0\}$, we get that $TT^+\zeta = \zeta$ and $\bar{T}T^+\zeta = 0$ and so $T^+\zeta \in N(\bar{T}) \cap R(T^+)$. Now from the assumption that $N(\bar{T}) \cap R(T^+) = \{0\}$, we obtain that $T^+\zeta = 0$. Thus, $\zeta = TT^+\zeta = 0$.

Corollary 3.2. *Let $T \in C(X,Y)$ with $T^+ \in D(Y,X)$ and let $\delta T \colon D(\delta T) \to D(T^+)$ be a linear operator such that $D(T) \subset D(\delta T)$ and δT is T–bounded with $b < 1$. Put $\bar{T} = T + \delta T \in C(X,Y)$.*

(1) *If $\bar{T}$ and T satisfy following conditions:*

$$\begin{aligned} N(\bar{T}) \cap R(T^+) &= \{0\}, & R(\bar{T}) \cap N(T^+) &= \{0\}, \\ D(T) &= N(\bar{T}) + R(T^+), & D(T^+) &= N(T^+) + R(\bar{T}), \end{aligned}$$

then $I_Y + \delta TT^+ \colon D(T^+) \to D(T^+)$ is bijective.

(2) *If $I_Y + \delta TT^+ \colon D(T^+) \to D(T^+)$ is bijective and $R(\bar{T}) \cap N(T^+) = \{0\}$, then $D(T) = N(\bar{T}) + R(T^+)$ and $D(T^+) = N(T^+) + R(\bar{T})$.*

Proof. (1) $R(\bar{T}) \cap N(T^+) = \{0\}$ implies that $\bar{T}R(T^+) \cap N(T^+) = \{0\}$. Since $D(T) = N(\bar{T}) + R(T^+)$, we have $R(\bar{T}) = \bar{T}R(T^+)$. Thus,

$$D(T^+) = R(\bar{T}) + N(T^+) = \bar{T}R(T^+) + N(T^+)$$

and hence $I_Y + \delta TT^+ \colon D(T^+) \to D(T^+)$ is bijective by Theorem 3.1.

(2) By Theorem 3.1, $D(T^+) = \bar{T}R(T^+) + N(T^+)$ when $I_Y + \delta TT^+$ is bijective. Noting that $\bar{T}R(T^+) \subset R(\bar{T}) \subset D(T^+)$, we have $D(T^+) = N(T) + R(\bar{T})$.

Since $I_X + T^+\delta T = I_X - T^+T + T^+\bar{T}$ is bijective by Lemma 2.4 and $(I_X + T^+\delta T)T^+ = T^+(I_Y + \delta TT^+)$ on $D(T^+)$, we have

$$I_X = (I_X + T^+\delta T)^{-1}(I_X - T^+T) + T^+(I_Y + \delta TT^+)^{-1}\bar{T} \quad \text{on } D(T).$$

Therefore, $D(T) = N(\bar{T}) + R(T^+)$ by Lemma 2.5.

Now we present our main result of the paper as follows.

Theorem 3.3. *Let X, Y be Banach Spaces and let $T \in C(X,Y)$ with $T^+ \in D(Y,X)$. Let $\delta T\colon D(\delta T) \to D(T^+)$ be a linear operator such that $D(T) \subset D(\delta T)$. Assume that δT is T–bounded with $b<1$ and $I_Y + \delta T T^+\colon D(T^+) \to D(T^+)$ is bijective. Put $\bar{T} = T + \delta T$ and $G = T^+(I_Y + \delta T T^+)^{-1}$. Consider following two statements* (A) and (B). We have

(A) The following conditions are equivalent:

(1) $R(\bar{T}) \cap N(T^+) = \{0\}$;

(2) $G = \bar{T}^+ \in D(Y,X)$ with $R(G) = R(T^+)$, $N(G) = N(T^+)$;

(3) $(I_Y + \delta T T^+)^{-1}\bar{T}$ maps $N(T)$ into $R(T)$;

(4) $(I_Y + \delta T T^+)^{-1} R(\bar{T}) = R(T)$;

(5) $(I_X + T^+\delta T)^{-1} N(T) = N(\bar{T})$.

(B) Further assume that $\delta T \in C(X,Y)$, $T^+ \in C(Y,X)$ and

$$c = \sup\{\|TT^+x\|| \, x \in D(T^+),\ \|x\| = 1\} < +\infty, \tag{3.1}$$

(e.g. T satisfies conditions of Proposition 2.2 (1)). If $bc < 1$ (note that $c \geq 1$), then $G \in C(Y,X)$.

Proof. We first prove statement (A).

(1) $\Rightarrow$ (2) We have $\bar{T} \in C(X,Y)$ and $G = T^+(I_Y + \delta T T^+)^{-1} = (I_X + T^+\delta T)^{-1}T^+$ by Lemma 2.4.

We now check that $\bar{T}G\bar{T} = \bar{T}$ on $D(T)$ and $G\bar{T}G = G$ on $D(T^+)$. We have

$$\begin{aligned}
\bar{T}G\bar{T} &= (T+\delta T)T^+(I_Y + \delta T T^+)^{-1}(T + \delta T)\\
&= (T+\delta T)(I_X + T^+\delta T)^{-1}(T^+T + T^+\delta T)\\
&= (T+\delta T)(I_X + T^+\delta T)^{-1}(T^+T - I_X + I_X + T^+\delta T)\\
&= -\bar{T}(I_X + T^+\delta T)^{-1}(I_X - T^+T) + \bar{T}\\
&= \bar{T}
\end{aligned}$$

on $D(T)$ by Lemma 2.5 since $R(\bar{T}) \cap N(T^+) = \{0\}$. Also, we have

$$\begin{aligned}
G\bar{T}Gy &= T^+(I_Y + \delta T T^+)^{-1}(T+\delta T)T^+(I_Y + \delta T T^+)^{-1}y\\
&= T^+(I_Y + \delta T T^+)^{-1}(I_Y + \delta T T^+)TT^+(I_Y + \delta T T^+)^{-1}y\\
&= T^+(I_Y + \delta T T^+)^{-1}y = Gy
\end{aligned}$$

for any $y \in D(T^+)$.

From $G = T^+(I_Y + \delta T T^+)^{-1} = (I_X + T^+\delta T)^{-1}T^+$, we obtain $R(G) = R(T^+)$ and $N(G) = N(T^+)$.

(2) $\Rightarrow$ (3) According to the proof of (1) $\Rightarrow$ (2), we have

$$\bar{T}(I_X + T^+\delta T)^{-1}(I_X - T^+T) = 0. \tag{3.2}$$

Thus, by (3.2),

$$(I_Y - TT^+)(I_Y + \delta TT^+)\bar{T}(I_X - T^+T) = (I_Y - TT^+)\delta T(I_X + T^+\delta T)(I_X - T^+T) = 0$$

on $D(T)$. This means that $(I_Y + \delta TT^+)^{-1}\bar{T}$ maps $N(T)$ into $R(T)$.

(3) $\Rightarrow$ (4) Let $x \in D(T)$ and put $x_1 = T^+Tx$, $x_2 = (I_X - T^+T)x \in N(T)$. Then $(I_Y + \delta TT^+)^{-1}\bar{T}x_2 \in R(T)$ by the assumption. Since

$$(I_Y + \delta TT^+)^{-1}\bar{T}x_1 = (I_Y + \delta TT^+)^{-1}(I_Y + \delta TT^+)Tx_1 = Tx_1 \in R(T),$$

it follows that $(I_Y + \delta TT^+)^{-1}R(\bar{T}) \subset R(T)$. On the other hand, for any $x \in D(T)$

$$(I_Y + \delta TT^+)Tx = \bar{T}T^+Tx \in R(\bar{T}) \subset D(T^+).$$

So $R(T) \subset (I_Y + \delta TT^+)^{-1}R(\bar{T})$ and consequently, $R(T) = (I_Y + \delta TT^+)^{-1}R(\bar{T})$.

(4) $\Rightarrow$ (1) Let $\xi \in R(\bar{T}) \cap N(T^+)$. Then $T^+\xi = 0$ and $\xi = (I_Y + \delta TT^+)T\eta$ for some $\eta \in D(T)$. Thus, $(I_X + T^+\delta T)T^+T\eta = 0$ and hence $T^+T\eta = 0$. This implies that $\xi = 0$.

The implication (1) $\Rightarrow$ (5) is Lemma 2.5. To complete the proof, we now show the implication (5) $\Rightarrow$ (3). Since $\bar{T}(I_X + T^+\delta T)^{-1}(I_X - T^+T) = 0$, we have

$$\begin{aligned} T(I_X + T^+\delta T)^{-1}(I_X - T^+T) &= -(I_Y + \delta TT^+)^{-1}\delta T(I_X - T^+T) \\ &= -(I_Y + \delta TT^+)^{-1}\bar{T}(I_X - T^+T), \end{aligned}$$

that is, $(I_Y + \delta TT^+)^{-1}\bar{T}$ maps $N(T)$ into $R(T)$.

(B) To prove $G \in C(Y,X)$, let $\{y_n\}_{n=1}^{\infty} \subset D(T^+)$ and $y \in Y$, $x \in X$ such that $\|y_n - y\| \to 0$ and $\|Gy_n - x\| \to 0$ $(n \to \infty)$. Set $z_n = (I_Y + \delta TT^+)^{-1}y_n \in D(T^+)$, $n \geq 1$. Then $z_n = y_n - \delta TT^+z_n$, $n \geq 1$ and $\|T^+z_n - x\| \to 0$ $(n \to \infty)$. Since δT is T–bounded, we have, for any m, $n \geq 1$,

$$\begin{aligned} \|z_n - z_m\| &\leq \|y_n - y_m\| + \|\delta TT^+(z_n - z_m)\| \\ &\leq \|y_n - y_m\| + a\|T^+z_n - T^+z_m\| + b\|TT^+(z_n - z_m)\| \\ &\leq \|y_n - y_m\| + a\|T^+z_n - T^+z_m\| + bc\|z_n - z_m\|. \end{aligned}$$

Thus, $\|z_n - z_m\| < (1 - bc)^{-1}(\|y_n - y_m\| + a\|T^+z_n - T^+z_m\|)$, m, $n \geq 1$ and that $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. Let $\|z_n - z\| \to 0$ as $n \to \infty$ for some $z \in Y$. Since $T^+ \in C(Y,X)$, it follows that $z \in D(T^+)$ and $T^+z = x$. From $\delta TT^+z_n = y_n - z_n \xrightarrow{\|\cdot\|} y - z$, $T^+z_n \xrightarrow{\|\cdot\|} x$ and $\delta T \in C(X,Y)$, we get that $x \in D(\delta T)$ and $\delta Tx = y - z$. Thus $y \in D(T^+)$, $x = T^+(y - \delta Tx)$ and hence $x = (I_X + T^+\delta T)^{-1}T^+y = Gy$.

Remark 3.4. Let $T \in C(X,Y)$ such that $T^+ \in C(Y,X)$ exists and let $\delta T \in B(X,Y)$ with $R(\delta T) \subset D(T^+)$. In this case, we do not need Condition (3.1). Put $\bar{T} = T + \delta T$. Then $\bar{T} \in C(X,Y)$ and $T^+\delta T \in B(X,X)$ by Closed Graph Theorem. Assume that $I_Y + \delta TT^+ \colon D(T^+) \to D(T^+)$ is bijective and $R(\bar{T}) \cap N(T^+) = \{0\}$. Then $G = (I_X + T^+\delta T)^{-1}T^+ \in C(Y,Y)$.

In fact, let $y \in Y$ and $x \in X$ and suppose that there is a sequence $\{y_n\}$ in Y such that $\|y_n - y\| \to 0$ and $\|Gy_n - x\| \to 0$ $(n \to \infty)$. Then

$$T^+y_n = (I_X + T^+\delta T)(I_X + T^+\delta T)^{-1}T^+y_n \xrightarrow{\|\cdot\|} (I_X + T^+\delta T)x.$$

Since $T^+ \in C(Y,X)$, we get that $y \in D(T^+)$ and $T^+y = (I_X + T^+\delta T)x$. Consequently, $Gy = x$. Therefore, $\bar{T}^+ = T^+(I_Y + \delta TT^+)^{-1} \in C(Y,X)$ by Theorem 3.3 (A).

In addition, if $T^+ \in B(Y,X)$, the results of Theorem 3.3 (A) are contained in [14, Chapter 2].

Remark 3.5. Let $T \in C(X,Y)$ with $T^+ \in B(Y,X)$ and let $\delta T\colon D(\delta T) \to Y$ be a T–bounded linear operator with $b < 1$ and $D(T) \subset D(\delta T)$. Then $\delta TT^+ \in B(Y,X)$. Suppose that $I_Y + \delta TT^+$ is invertible in $B(Y,Y)$ and $R(T+\delta T) \cap N(T^+) = \{0\}$. Then the bounded linear operator $T^+(I_Y + \delta TT^+)^{-1}$ is a generalized inverse of $T + \delta T$ by Theorem 3.3. This result is Theorem 2.1 of [7]. However, in this case, the equivalence of the conditions (1)—(5) of Theorem 3.3 (A) is not given in [7].

In addition, if there are constants a, $b > 0$ such that

$$a\|T^+\| + b\|TT^+\| < 1, \ \|\delta Tx\| \leq a\|x\| + b\|Tx\|, \ \forall\, x \in D(T),$$

then $\|\delta TT^+\| < 1$ and $b < 1$ for $\|TT^+\| \geq 1$. Thus, $\bar{T}$ is a closed operator and $I_Y + \delta TT^+$ is invertible in $B(Y,Y)$. Therefore, the conditions (1)—(5) of Theorem 3.3 (A) are equivalent. This result is Theorem 2.1 in [8].

Finally, combining Proposition 2.2 (2) with Theorem 3.3 (A), we have

Corollary 3.6. *Let $T \in C(X,Y)$ with $R(T)$ closed such that $T^+_{P,Q}$ exists. Let $\delta T \in B(X,Y)$ such that $I_X + T^+_{P,Q}\delta T$ is invertible in $B(X,X)$ and $R(T+\delta T) \cap N(T^+_{P,Q}) = \{0\}$. Then $R(T+\delta T)$ is closed and $(T+\delta T)^+_{\bar{P},\bar{Q}} = (I_X + T^+_{P,Q}\delta T)^{-1}T^+_{P,Q}$, where $\bar{P} = (I_X + T^+_{P,Q}\delta T)^{-1}P(I_X + T^+_{P,Q}\delta T)$ and $\bar{Q} = (I_Y + \delta TT^+_{P,Q})T^+_{P,Q}(I_Y + \delta TT^+_{P,Q})^{-1}$.*

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